MA 734 Exam 2 Review

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28 March 2023

Definitions

Periodic in D

Let $w = f(z)$ be a single-valued function defined in the domain D of the complex plane, and let $\lambda \neq 0$ be a constant complex number. Suppose that for every $z \in D$, $z + \lambda \in D$. The function $f(z)$ is said to be **periodic in D**, if for all $z \in D$, $f(z + \lambda) = f(z)$.

Example: Since $e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z \ \forall z \in \mathbb{C}$, e^z is periodic in $\mathbb{C}$.

Fundamental Period

We call the complex number $\lambda = 2\pi i$ the **fundamental period** of the function e^z in the sense that any other period w of e^z must be of the form $2\pi k i$, where $k \in \mathbf{Z} \setminus \{0\}$.
Example: Since $e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z \ \forall z \in \mathbb{C}$, e^z is periodic in $\mathbb{C}$.

(Principle) Logarithm

Let $z \neq 0, z \in \mathbb{C}$. If $w \in \mathbb{C}$ and $e^w = z$, then w is called a **logarithm** of z where $w = \log z$. This is a **set-valued** function where

$$w = \log z = \{\ln|z| + i \operatorname{Arg} z + 2n\pi i \mid n \in \mathbb{Z}\}, \text{ for } -\pi < \operatorname{Arg} z \leq \pi$$

Note that $w = ln|z| + i \operatorname{Arg} z$, that is $n = 0$, is called the **principle logarithm** and is denoted by $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$.

(Principle) Branch

A **branch** is a restricted range of a multi-valued function $w = f(z)$ over which there is a single value w for each z and for which $f(z)$ is continuous. If the branch's range is restricted to the principle values, we call it the **principle branch** of the function.

Branch Cut

A **branch cut** of a multi-valued function is a curve in the complex plane across which the function is discontinuous. A branch cut is used to separate a multi-valued function into single-valued and continuous parts (branches).

Branch Point

The point z_0 is called a **branch point** of a complex multi-valued function $f(z)$ if it is shared by multiple branch cuts of the function.

Complex Exponent

If $z \neq 0$ and $c \in \mathbb{C}$, we define z^c by

$$z^c = e^{c \log z}$$

where $\log z$ is the multi-valued logarithm of z. The principle value of z^c is defined by

$$z^c = e^{c \operatorname{Log} c}$$

Trigonometric Functions

For all $z \in \mathbb{C}$:

$$\begin{aligned}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\
\sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\
\frac{d}{dz}(\cos z) &= \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = -\sin z \\
\frac{d}{dz}(\sin z) &= \frac{d}{dz}\left(\frac{e^{-iz} - e^{iz}}{2i}\right) = \cos z
\end{aligned}$$

Not that all trig. identities involving sin and cos over $\mathbb{R}$ hold over to $\mathbb{C}$ (see Theorem. 2.27.A).

Inverse Trigonometric Functions

For all $z \in \mathbb{C}$:

$$\begin{aligned}
\sin^{-1} z &= -i \log\left[iz + (1 - z^2)^{\frac{1}{2}}\right] \\
\cos^{-1} z &= -i \log\left[z + i(1 - z^2)^{\frac{1}{2}}\right] \\
\tan^{-1} z &= (\frac{i}{2}) \log\left(\frac{i + z}{i - z}\right)
\end{aligned}$$

Given these are defined in terms of logarithms and square roots, all these functions are multi-valued.

If a specific branch is chosen for these functions, they become analytic and have the following derivatives:

$$
\begin{aligned}
\frac{d}{dz}(\sin^{-1} z) &= \frac{1}{\sqrt{1-z^2}} \\
\frac{d}{dz}(\cos^{-1} z) &= -\frac{1}{\sqrt{1-z^2}} \\
\frac{d}{dz}(\tan^{-1} z) &= \frac{1}{1-z^2}
\end{aligned}
$$

Derivatives of Functions $w(t)$

Consider a complex valued function of a single real variable t. For $t \in [a,b]$,

$$w(t) = U(t) + iv(t)$$

Define $w'(t)$ as

$$
\begin{aligned}
w'(t) &= \lim_{\Delta t \to 0} \frac{w(t+\Delta t) - w(t)}{\Delta t} \\
&= u'(t) + iv'(t)
\end{aligned}
$$

provided all the derivatives exists.

Definite Integrals of Functions $w(t)$

For $w(t) = u(t) + iv(t), t \in [a,b]$ we define the integral of $w(t)$ over $[a,b]$ as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the integrals on the RHS exist. Now, if $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + V(t)$ are continuous on the interval $[a,b]$ and $W'(t) = w(t)$ $\forall t \in [a,b]$, then

$$
\begin{aligned}
\int_a^b w(t)dt &= \int_a^b u(t)dt + i\int_a^b v(t)dt \\
&= [U(t)]_a^b + i[V(t)]_a^b \\
&= (U(b) + iV(b)) - (U(a) + iV(a)) \\
&= W(b) - W(a)
\end{aligned}
$$

Note that the Mean Value Theorem does not hold for complex integrals.

Arcs, Curves, and Contours

1. A continuous function of a real variable t, $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{C}$ is called an **arc**. We can express $\gamma(t)$ as

$$\gamma(t) = x(t) + iy(t)]$$

for $t \in [a,b]$, where $x(t)$ and $y(t)$ are real valued functions and are called the real and imaginary parts of γ. We denote the arc $\gamma(t)$ by $\mathcal{C}$.

2. We say arc $\mathcal{C} : \gamma(t)$ for $t \in [a,b]$ is **simple** if γ is injective (i.e. the curve does not cross itself.

3. A **closed curve** is an arc $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma(a) = \gamma(b)$.

4. A closed curve $\gamma : [a,b] \to \mathbb{C}$ is said to be simple when γ is injective on the open interval (a,b) (it is a **simple closed curve**).

5. We say $\gamma : [a,b] \to \mathbb{C}$ is a **differentiable arc** when γ' is defined and continuous on $[a,b]$. If we *additionally* have that $z'(t) \neq 0$ for all $t \in (a,b)$, then we say γ is a **smooth arc**.

6. If $\mathcal{C}$ is the arc $\gamma : [a,b] \to \mathbb{C}$, then we write $-\mathcal{C}$ for the arc that we denote by $\gamma^- : [-b,-a] \to \mathbb{C}$ and that is given by

$$\gamma^-(t) = \gamma(-t)$$

for $-b \leq t \leq -a$. Put simply, this is the same arc as γ, but traversed *backwards*. We call $-\mathcal{C}$ the **opposite arc** of $\mathcal{C}$.

Alternative definition: We may define γ^- as

$$\gamma^- : [0,1] \to \mathbb{C}, \text{ where}$$
$$\gamma^-(t) = \gamma(0 + 1 - t) = \gamma(1-t)$$

Note that in this case, $\gamma^-(0) = \gamma(1) = i$ and $\gamma^-(1) = \gamma(0) = 1$.

7. A **contour** is a piecewise smooth arc. Simply put, a contour is an arc that is created by placing finitely many smooth arcs end to end. As a function, this is defined as $\gamma : [a,b] \to \mathbb{C}$ where

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } a = a_0 \leq t \leq a_1 \\ \gamma_2(t) & \text{if } a = a_1 \leq t \leq a_2 \\ \vdots & \\ \gamma_n(t) & \text{if } a = a_{n-1} \leq t \leq a_n = b \end{cases}$$

where $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and

$$\gamma_1(a_1) = \gamma_2(a_2), \ldots, \gamma_{n-1}(a_{n-1}) = \gamma_n(a_{n-1})$$

We call $\gamma_1, \gamma_2, \dots, \gamma_n$ parametrizations of smooth arcs $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$. In this case we write

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_n$$

Note: There is no unique parametrization for a contour $\mathcal{C}$.

Curve Orientation

A simple closed curve is said to be **positively oriented** when it is traced out in a counter-clockwise direction as t ranges from a to b.

It is said to be **negatively oriented** when it is traced out in a clockwise direction as t ranges from a to b.

Reparametrizations

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. A curve $\widetilde{\gamma} : [\alpha, \beta] \to \mathbb{C}$ is called a **reparametrization** of γ if there is a $\mathcal{C}^1$ function $\varphi : [a, b] \to [\alpha, \beta]$ with $\varphi'(t) > 0$, $\varphi(a) = \alpha$, and $\varphi(b) = \beta$ such that

$$\gamma(t) = (\widetilde{\gamma} \circ \varphi)(t) = \widetilde{\gamma}(\varphi(t))$$

That is to say, there is **no unique parametrization** for a contour. Note that the contour integral and arc length of a complex valued function is independent of the parametrization of the contour we integrate over.

Contour Integral

Suppose $f : D \to \mathbb{C}$ is acomplex valued function, where $D \subset \mathbb{C}$. Let $\mathcal{C}$ be an arc given by the function $\gamma : [a, b] \to D$ in D. f is said to be continuous on arc $\mathcal{C}$ if the function $\varphi(t) = (f \circ \gamma)(t)$ is continuous on $[a, b]$. In this case, the integral of $f(z)$ along the arc $\mathcal{C}$ is denoted by $\int_\gamma f(z)dz$ and it is defined by

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b (f \circ \gamma)(t)\gamma'(t)dt \\ &= \int_a^b f(\gamma(t))\gamma'(t)dt\end{aligned}$$

where the integral on the right hand side is the Riemann integral of a real variable t.

Initial & Terminal Points

Suppose that $a = a_1 < b_1 = a_2 < b_2, \cdots = a_k < b_k = b$ are real numbers, and that for every $j = 1, 2, \dots, k$, $\mathcal{C}_j$ is an arc given by $\gamma_j : [a_j, b_j] \to D \subset \mathbb{C}$. Suppose further that $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}$ for every $j = 1, \dots, k-1$. Then, $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_k$ is called a contour. The point $\gamma_1(a_1)$ is called the **initial**

point of the contour $\mathcal{C}$, and the point $\gamma_k(b_k)$ is called the **terminal point** of the counter $\mathcal{C}$. A complex valued function $f : D \to \mathbb{C}$ is said to be **continuous on the contour** $\mathcal{C}$ if it is continuous on the arc C_j for every $j = 1, 2, \ldots, k$. In this case, the integral of f along $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}_1} f + \int_{\mathcal{C}_2} f + \cdots + \int_{\mathcal{C}_k} f$$

Simply Connected Domains

Informally, **simply connected domain** is an open connected set with "no holes." An extension of Cauchy-Goursat tells us that the integral of a function that is analytic over a simply connected domain is 0 for all closed contours in the domain.

A more formal definition follows: A simply connected domain D is a domain such that every simple contour in the domain encloses only points in D. Another way to view this is that a domain is simply connected if any simple closed contour $\mathcal{C}$ lying entirely in D can be shrunk to a point without leaving D.

Multiply Connected Domain

A domain that is not simply connected is called a **multiply connected domain**. Thus a multiply connected domain has "holes" in it.

Theorems

3.29.A

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then $e^{z_1}e^{z_2} = e^{z_1+z_2}$. Note if $z_1 = z$ and $z_2 = -z$, then $e^z \cdot e^{-z} = e^0 = 1$, therefore $e^{-z} = \frac{1}{e^z}$.

Proof:

$$\begin{aligned}
e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) \cdot e^{x_2}(\cos y_2 + i \sin y_2) \\
&= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_2 \sin y_2) \\
&+ i(\sin y_1 \cos y_2 + \sin y_2 \cos y_2)] \\
&= e^{x_1+x_2}(\cos y_1 + y_2 + i \sin y_1 + y_2) \\
&= e^{z_1+z_2}
\end{aligned}$$

$\square$

3.29.1

(a) $\forall z \in \mathbb{C}$, $e^z \neq 0$.

(b) $|e^{iy}| = 1$ and $|e^z| = e^x$ $\forall z = x + iy \in \mathbb{C}$.

(c) A necessary and sufficient condition that $e^z = 1$ is $z = 2\pi k i, k \in \mathbb{Z}$.

(d) For $z_1, z_2 \in \mathbb{C}$, we have $e^{z_1} = e^{z_2}$ if and only if $z_1 = z_2 \pmod{2\pi i}$.

Proof:

(a) By Lemma 3.29.A we have:

$$e^z e^{-z} = e^{z+(-z)} = e^0 = 1$$

Since the product is never zero, neither factor can be zero. Therefore, $e^z \neq 0$.

(b)

$$e^{iy} = \cos y + i \sin y$$
$$|e^{iy}| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

It follows,

$$|e^z| = |e^x e^{iy}| = |e^x||e^{iy}| = |e^x| = e^x$$

(c) Suppose that $e^z = 1$. Thus $e^z = e^x \cos y + ie^x \sin y = 1 + 0i$. This implies $e^x \cos y = 1$ and $e^x \sin y = 0$. Since $e^x \neq 0 \Rightarrow \sin y = 0 \Rightarrow y = n\pi$ for some $n \in \mathbb{Z}$. But $\cos n\pi = (-1)^n$. Since $e^x > 0$ we see that $e^x(-1)^n = 1$ only if $x = 0$ and $n = 2k$ for some $k \in \mathbb{Z}$. Thus $z = x + iy = 0 + in\pi = 2\pi k i$.

Conversely, assume that $z = 2\pi k i$, where $k \in \mathbb{Z}$. Then by the definition of the exponential function,

$$e^z = e^{2\pi k i} = \cos 2\pi k + i \sin 2\pi k = 1$$

(d) $e^{z_1} = e^{z_2}$ if and only if $\frac{e^{z_1}}{e^{z_2}} = 1$.
$\Rightarrow e^{z_1 - z_2} = 1 \iff z_1 - z_2 = 2\pi k i$
$\Rightarrow z_1 \equiv z_2 \pmod{2\pi i}$

□

Claim 1

The exponential function defined for every $z \in R_0$ by $f(z) = e^z$ is a bijection.

Proof:
Assume for $z_1, z_2 \in R_0$ we have

$$\begin{aligned} e^{z_1} = e^{z_2} &\Rightarrow z_1 \equiv z_2 \pmod{2\pi i} \\ &\Rightarrow z_1 - z_2 = 2\pi k i \end{aligned}$$

for some $k \in \mathbb{Z}$. But since $z_1, z_2 \in R_0$, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
-\pi < y_1, y_2 \le \pi \Rightarrow -2\pi < y_1 - y_2 < 2\pi \\
\Rightarrow k = 0 \\
\Rightarrow z_1 = z_2
\end{aligned}$$

Therefore, $f(z)$ is one-to-one.

To show $f(z) = e^z$ is onto, let $w = e^z$, where $w \neq 0$ is given, and $z = x + iy$ be unknown.

$$\begin{gathered}
|w| = e^x \Rightarrow x = \ln |w| \\
\arg w = y + 2\pi k \text{ or } y = \arg w
\end{gathered}$$

Obviously, there are infinitely many values of w, since $\arg w$ takes infinitely many values, all differing by integer multiples of 2π. Exactly, one of these values corresponds to a unique $z \in R_0$. $\square$

3.29.2

The exponential function $f(z) = e^z$ is analytic for all $z \in \mathbb{C}$. Moreover, $f'(z) = e^z$.

Proof:
Let $f(z) = e^z = u(x, y) + iv(x, y)$. This implies $f(z) = e^x \cos y + ie^x \sin y \Rightarrow u(x, y) = e^x \cos y, v(x, y) = e^x \sin y$. So,

$$\begin{aligned}
\frac{du}{dx} &= e^x \cos y \\
\frac{du}{dy} &= -e^x \sin y \\
\frac{dv}{dx} &= e^x \sin y \\
\frac{dv}{dy} &= e^x \cos y
\end{aligned}$$

Hence, $\frac{du}{dx} = \frac{dv}{dy}$ and $\frac{du}{dy} = -\frac{dv}{dx}$. Thus u and v satisfy the C-R equations and since these partial derivatives are continuous $\forall (x, y) \in \mathbb{R}^2$, then $f(z) = e^z$ is analytic $\forall z \in \mathbb{C}$. Moreover,

$$\begin{aligned}
f'(z) &= \frac{du}{dx} + i\frac{dv}{x} \\
&= e^x \cos y + ie^x \sin y \\
&= e^z
\end{aligned}$$

$\square$

3.30.1

For any complex number $z \neq 0$, there exists some complex numbers w such that $e^w = z$. In particular, one such w is the complex number $\operatorname{Log} z$, defined as

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

This is called the **principle log**. Other values of w are given by $\{\ln|z| + i \operatorname{Arg} z + 2n\pi i \mid n \in \mathbb{Z}\}$ which is denoted by $\log z$.

Proof:
Writing $z = x + iy$ in polar exponential form: $z = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \operatorname{Arg} z$ for $-\pi < \theta \leq \pi$. Now, observe that

$$\begin{aligned} e^{\ln|z| + i \operatorname{Arg} z} &= e^{\ln|z} \cdot e^{i \operatorname{Arg} z} \\ &= |z| e^{i\theta} \\ &= z \end{aligned}$$

Hence, $w = \ln|z| + i \operatorname{Arg} z$ is a solution of the equation $e^w = z$. Now suppose that w_1 is another solution of $e^w = z$. Then,

$$\begin{aligned} & e^{w_1} = e^w = z \\ & \Rightarrow e^{w_1 - w} = 1 \\ & \Rightarrow w_1 = w \pmod{2\pi i} \\ & \Rightarrow w_1 - w = 2n\pi i, \text{ for some } n \in \mathbb{Z} \\ & \Rightarrow w_1 = w + 2n\pi i \\ & \Rightarrow w_1 \in \{\ln|z| + i \operatorname{Arg} z + 2n\pi i \mid n \in \mathbb{Z}\} \end{aligned}$$

□

Remarks 1

Fix $\alpha \in \mathbb{R}$ and define

$$\log z = \ln r + i\theta$$

where $z = re^{i\theta}$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$. This defines a branch of the logarithm with a branch cut $\theta = \alpha$.

1. The function $z \mapsto \log z$ is analytic.
2. When $\alpha = -\pi$, we call $\log z$ the principle branch of the logarithm and we denote it by $\operatorname{Log} z$.
3. Let $\log z$ be any branch of the logarithm. Then,

$$\frac{d}{dz}(\log z) = \frac{1}{z}$$

Proof:

1. $f(z) = \log z = \ln r + i\theta$ implies
$$u(r,\theta) = \ln r, v(r,\theta) = \theta$$
$$\frac{du}{dr} = \frac{1}{r}, \frac{du}{d\theta} = 0$$
$$\frac{dv}{dr} = 0, \frac{dv}{d\theta} = 1$$
The C-R equations in polar coordinates
$$\frac{du}{dr} = \frac{1}{r} = \frac{1}{r}\frac{dv}{d\theta} = \frac{1}{r} \cdot 1$$
$$\frac{dv}{dr} = 0 = -\frac{1}{v}\frac{du}{d\theta} = (-\frac{1}{r})(0)$$
are satisfied. Moreover $\frac{du}{dr}, \frac{du}{d\theta}, \frac{dv}{dr}$, and $\frac{dv}{d\theta}$ are continuous. Therefore $f(z) = \log z$ is analytic.

2. Not necessary.

3. We have $e^{\log z} = z$. This implies
$$\frac{d}{dz}(e^{\log z}) = \frac{d}{dz}(z)$$
$$e^{\log z}\frac{d}{dz}(\log z) = z\frac{d}{dz}(\log z) = 1$$
So, $\frac{d}{dz}(\log z) = \frac{1}{z}$.

□

3.32.A

For the set-valued (or multi-valued) logarithm, we have for all nonzero $z_1, z_2 \in \mathbb{C}$:
$$\log z_1 z_2 = \log z_1 + \log z_2$$
Note that the left and right sides represent infinite sets (same goes for 3.32.B and 3.32.C).

Proof:
By definition, $\log z = \ln|z| + i\arg z$, and by Lemma 1.8.1 $\arg z_1 z_2 = \arg z_1 + \arg z_2$, then
$$\begin{aligned}\log z_1 z_2 &= \ln|z_1 z_2| + i\arg z_1 z_2 \\ &= \ln|z_1| + \ln|z_2| + i\arg z_1 + i\arg z_2 \\ &= (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2) \\ &= \log z_1 + \log z_2\end{aligned}$$

□

3.32.B

For the set-valued (or multi-valued) logarithm, we have for all nonzero $z_1, z_2 \in \mathbb{C}$:

$$\log \frac{z_1}{z_2} = \log z_1 - \log z_2$$

Proof:

$$\begin{aligned}
\log \frac{z_1}{z_2} &= \ln \left|\frac{z_1}{z_2}\right| + i \arg \frac{z_1}{z_2} \\
&= \ln |z_1| - \ln |z_2| + i \arg z_1 - i \arg z_2 \\
&= (\ln |z_1| + i \arg z_1) - (\ln |z_2| + i \arg z_2) \\
&= \log z_1 - \log z_2
\end{aligned}$$

$\square$

3.32.C

If $z \in \mathbb{C}$ is nonzero and n is an integer, then $\log z^n = n \log z$.

Proof:

$$\begin{aligned}
\log z^n &= \ln |z^n| + i \arg z^n \\
&= n \ln |z| + in \arg z \\
&= n(\ln |z| + i \arg z) \\
&= n \log z
\end{aligned}$$

$\square$

3.32.D

(a) For any nonzero $z \in \mathbb{C}$ and $n \in \mathbb{Z}$:

$$z^n = e^{n \log z}$$

(b) Let n be a positive intger, and let $z \in \mathbb{C} \setminus \{0\}$:

$$z^{\frac{1}{n}} = e^{\frac{1}{n} \log z}$$

Proof:

(a) Write $z = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$. Then $z^n = r^n e^{in\theta}$. For $k \in \mathbb{Z}$:

$$\begin{aligned}
e^{n \log z} &= e^{n(\ln r + i\theta + 2\pi k)} \\
&= e^{n \ln r + i(n\theta + 2\pi kn)} \\
&= e^{n \ln r} \cdot e^{i(n\theta + 2\pi kn)} \\
&= e^{n \ln r}[\cos(n\theta + 2\pi kn) + i \sin(n\theta + 2\pi kn)] \\
&= e^{\ln r^n}(\cos n\theta + i \sin n\theta) \\
&= r^n e^{in\theta} \\
&= z^n
\end{aligned}$$

(b) Write $z = re^{i\theta}, r > 0, \theta \in \mathbb{R}$. For $k \in \mathbb{Z}$:

$$\begin{aligned}
e^{\frac{1}{n} \log z} &= e^{\frac{1}{n}(\ln r + i(\theta + 2\pi k))} \\
&= e^{\frac{1}{n} \ln r} \cdot e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})} \\
&= e^{\ln r^{\frac{1}{n}}} \cdot e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})} \\
&= \sqrt[n]{r} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})}, k \in \{0, 1, 2, \ldots, n-1\} \\
&= z^{\frac{1}{n}}
\end{aligned}$$

$\square$

3.32.D Corollary

For any $z \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{Z}, n > 0$:

$$z^{\frac{m}{n}} = e^{(\frac{m}{n}) \log z}$$

Remarks 2

1. z^c is single-valued when c is a real integer.
2. z^c takes finitely many values when c is a real rational number.
3. z^c takes infinitely many values in all other cases.

Properties of Complex Exponents

Let $z \neq 0$, α, and β be complex numbers. Then:

1. $z^\alpha \cdot z^\beta = z^{\alpha+\beta}$
2. $\frac{z^\alpha}{z^\beta} = z^{\alpha-\beta}$
3. $(z^\alpha)^n = z^{n\alpha}$, $n \in \mathbb{Z}$

4. $(z_1 \cdot z_2)^\alpha \neq z_1^\alpha \cdot z_2^\alpha$ (in general)
5. $(z^\alpha)^\beta \neq z^{\alpha\beta}$ (in general)

Proof:

1. $$\begin{aligned} z^\alpha \cdot z^\beta &= e^{\alpha \log z} \cdot e^{\beta \log z} \\ &= e^{\alpha \log z + \beta \log z} \\ &= e^{(\alpha+\beta) \log z} \\ &= z^{\alpha+\beta} \end{aligned}$$

2. By (1) we have $z^{\alpha-\beta} \cdot z^\beta = z^{(\alpha-\beta)+\beta} = z^\alpha$. Divide both sides by z^β:
$$z^{\alpha-\beta} = \frac{z^\alpha}{z^\beta}$$

3. $$(z^\alpha)^n = (e^{\alpha \log z})^n = e^{(n\alpha) \log z} = z^{n\alpha}$$

4. Think of an example
5. Consider $z = i, \alpha = 4, \beta = \frac{1}{2}$.

□

Jordan Curve Theorem

If $\mathcal{S}$ is the range of a simple closed curve in the complex plane $\mathbb{C}$, then the complement $\mathbb{C} \setminus \mathcal{S}$ is the union of two disjoint domains, one g which is bounded and the other of which is unbounded.

Proof:
Here be dragons! □

Properties of Complex Integrals

1. The integral $\int_\mathcal{C} f(z)dz$ is a special type of line integral.
2. The following two properties imply integration of complex functions along an arc is a linear operation:
$$\int_\mathcal{C} (f(z) + g(z))\, dz = \int_\mathcal{C} f(z)dz + \int_\mathcal{C} g(z)dz$$
$$\int_\mathcal{C} cf(z)dz = c \int_\mathcal{C} f(z)dz, \text{ where } c \in \mathbb{C} \text{ is a constant.}$$

3. $\int_{-\mathcal{C}} f(z)dz = -\int_{\mathcal{C}} f(z)dz$

4. If $\widetilde{\gamma}$ is a reparametrization of γ, then

$$\int_{\gamma} f(z)dz = \int_{\widetilde{\gamma}} f(z)dz$$

for any continuous f defined on an open set containing the image of $\gamma =$ image of $\widetilde{\gamma}$.

5. Given a contour $\mathcal{C}$ such that $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_k$, and a function f that is continuous along the contour,

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}_1} f + \int_{\mathcal{C}_2} f + \cdots + \int_{\mathcal{C}_k} f$$

Proof:

1. Let $\mathcal{C} : \gamma(t) = x(t) + iy(t)$, where $a \le t \le b$. And let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.

$$\begin{aligned}
\int_{\mathcal{C}} f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b (u(\gamma(t)) + iv(\gamma(t)))\gamma'(t)dt \\
&= \int_a^b (u(\gamma(t)) + iv(\gamma(t)))(x'(t) + iy'(t))dt \\
&= \int_a^b [u(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]\, dt \\
&= \int_a^b \left[u(x(t), y(t))\frac{dx}{dt}\right] dt - \int_a^b \left[v(x(t), y(t))\frac{dy}{dt}\right] dt \\
&+ i\int_a^b \left[v(x(t), y(t))\frac{dx}{dt}\right] dt + i\int_a^b \left[u(x(t), y(t))\frac{dy}{dt}\right] dt \\
&= \int_a^b udx - \int_a^b vdy + i\int_a^b vdx + i\int_a^b udy \\
&= \int_{\mathcal{C}} udx - vdy + i\int_{\mathcal{C}} vdx + udy
\end{aligned}$$

2. Exercise.

3. Let $\mathcal{C}$ be given by $\gamma : [a, b] \to D$, then the opposite arc $-\mathcal{C}$ is given by $\gamma^- : [a, b] \to \mathbb{C}$ and defined by

$$\gamma^-(t) = \gamma(a + b - t), t \in [a, b]$$

Now,

$$\begin{aligned}
\int_{-\mathcal{C}} f(z)dz &= \int_{\gamma^-} f(z)dz \\
&= \int_a^b \left(f \circ \gamma^-\right)(t)\left(\gamma^-(t)\right)' dt \\
&= \int_a^b f\left(\gamma(a+b-t)\right)\frac{d}{dt}\left(\gamma(a+b-t\right) dt \\
&= \int_a^b f\left(\gamma(a+b-t)\right)\gamma'(a+b-t)(-1)dt \\
&= -\int_a^b f\left(\gamma(a+b-t)\right)\gamma'(a+b-t)dt
\end{aligned}$$

Substituting $u = a+b-t$ we obtain

$$\begin{aligned}
-\int_b^a f\left(\gamma(u)\right)\gamma'(u)(-du) &= \int_b^a f\left(\gamma(u)\right)\gamma'(u)du \\
&= -\int_a^b \left(f\left(\gamma(u)\right)\gamma'(u)du \right.\\
&= -\int_\gamma f(z)dz
\end{aligned}$$

Hence $\int_{-\mathcal{C}} f(z)dz = -\int_{\mathcal{C}} f(z)dz$.

4. Let $\gamma : [a,b] \to 0$ be a smooth arc, and let $\widetilde{\gamma} : [\alpha, \beta] \to D$ be a reparametrization of γ. Thus thee is a $\mathcal{C}^1$-function $\varphi : [a,b] \to [\alpha,\beta]$ with $\varphi'(t) > 0$, $\varphi(a) = \alpha$, and $\varphi(b) = \beta$ such that $\gamma(t) = \widetilde{\gamma}(\varphi(t))$.

$$\int_\gamma f = \int_a^b f\left(\gamma(t)\right)\cdot\gamma'(t)dt$$

By the chain rule,

$$\gamma'(t) = \frac{d}{dt}\left[\widetilde{\gamma}(\varphi(t))\right] = \widetilde{\gamma}'\left(\varphi(t)\right)\cdot\varphi'(t)$$

Let $u = \varphi(t)$ be a new variable so that $u = \alpha = \varphi(a)$ and $u = \beta = \varphi(b)$. Then,

$$\begin{aligned}
\int_a^b f\left(\gamma(t)\right)\gamma'(t)dt &= \int_a^b f\left(\widetilde{\gamma}(\varphi(t))\right)\widetilde{\gamma}'(\varphi(t))\frac{du}{dt}dt \\
&= \int_\alpha^\beta f\left(\widetilde{\gamma}(u)\right)\widetilde{\gamma}'(u)du \\
&= \int_{\widetilde{\gamma}} f
\end{aligned}$$

□

subsection*4.43.A If $w : [a, b] \to \mathbb{C}$ is a piecewise continuous complex valued function in the real closed interval $[a, b]$, then

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$$

Proof:
Let $I = \left|\int_a^b w(t)dt\right|$. If $I = 0$, then clearly

$$0 = \left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$$

If $I > 0$, then there exists a real number θ such that

$$\int_a^b w(t)dt = Ie^{i\theta}$$

Which implies

$$\begin{aligned} I &= e^{-i\theta} \int_a^b w(t)dt \\ &= \int_a^b e^{-i\theta} w(t)dt \\ &= \int_a^b \mathrm{Re}(e^{-i\theta} w(t))dt + i \int_a^b \mathrm{Im}(e^{-i\theta} w(t))dt \end{aligned}$$

Since $I = \left|\int_a^b w(t)dt\right|$ is a real number, then $\int_a^b \mathrm{Im}(e^{-i\theta} w(t))dt = 0$. Hence, $I = \int_a^b \mathrm{Re}(e^{-i\theta} w(t))dt$. On the other hand, clearly

$$\mathrm{Re}(e^{-i\theta} w(t)) \leq \left|e^{-i\theta} w(t)\right| = |w(t)|$$

for all $t \in [a, b]$. Thus,

$$\begin{aligned} I &= \int_a^b \mathrm{Re}\left(e^{-i\theta} w(t)\right) dt \\ &\leq \int_a^b |w(t)|dt. \end{aligned}$$

That is to say,

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt.$$

□

4.43.1

Let $\mathcal{C}$ be a contour of length L, and suppose that f is a piecewise continuous function on $\mathcal{C}$. If $M \geq 0$ is a constant such that $|f(z)| \leq M$ for all $z \in \mathcal{C}$, then $\left|\int_{\mathcal{C}} f(z)dz\right| \leq ML$.

Proof:
WLOG we may assume that $\mathcal{C}$ is a smooth arc given by the function $\gamma : [a, b] \to \mathbb{C}$. Then,

$$\begin{aligned}\left|\int_{\mathcal{C}} f(z)dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \\ &\leq \int_a^b |f(\gamma(t))\gamma'(t)|\, dt \\ &\leq \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt \\ &\leq \int_a^b M\, |\gamma'(t)|\, dt \\ &= M\int_a^b |\gamma'(t)|\, dt \\ = ML\end{aligned}$$

□

4.44.A

Let f be a continuous complex-valued function on the domain $D \subset \mathbb{C}$. Then the following statements are equivalent:

(a) The function f has an antiderivative on D.

(b) For any $z_1, z_2 \in D$ and contours $\mathcal{C}_1$ and $\mathcal{C}_2$ in D from z_1 to z_2,

$$\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$$

(c) For every closed contour $\mathcal{C}$ in D,

$$\int_{\mathcal{C}} f(z)dz = 0$$

Proof:
(a) ⇒ (b) First, we assume $\mathcal{C}$ is a smooth arc from z_1 to z_2 parametrized by

$\gamma : [a, b] \to \mathbb{C}$. Assume f has an antiderivative F on D. Then $F'(z) = f(z)$. Consider the composite function $F(\gamma(t))$, by the chain rule we have

$$\begin{aligned}
\frac{d}{dt}\left(F(\gamma(t))\right) &= F'(\gamma(t)) \cdot \gamma'(t) \\
&= f(\gamma(t)) \cdot \gamma'(t) \\
&\Rightarrow \int_{\mathcal{C}} f(z)dz \\
&= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b \frac{d}{dt}\left(F(\gamma(t))\right)dt \\
&= \left[F(\gamma(t))\right]_a^b \\
&= F(\gamma(b)) - F(\gamma(a)) \\
&= F(z_2) - F(z_1)
\end{aligned}$$

Now assume $\mathcal{C}$ is a contour from point z_1 to z_2. Then $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_n$, where $\mathcal{C}_i$ is a smooth arc, for all $1 \le i \le n$ with parametric representation given by $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$ such that $\gamma_1(a_0) = z_1$, $\gamma_n(a_n) = z_2$. Now,

$$\begin{aligned}
\int_{\mathcal{C}_\rangle} f(z)dz &= F(a_i) - F(a_{i-1}) \\
&\Rightarrow \int_{\mathcal{C}} f(z)dz \\
&= \sum_{i=1}^{n} \int_{\mathcal{C}_\rangle} f(z)dz \\
&= \sum_{i=1}^{n} (F(a_i) - F(a_{i-1})) \\
&= F(a_n) - F(a_0) \\
&= F(z_2) - F(z_1)
\end{aligned}$$

(b) $\Rightarrow$ (c) Suppose that the integral of $f(z)$ is independent of the contour in D and it only depends on the endpoints of the contour. Let $\mathcal{C}$ be any closed contour in D and let z_1 and z_2 be any two distinct points on $\mathcal{C}$. Form two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ from z_1 to z_2. Since the values of the integral of $f(z)$ is independent of the contours, then $\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$ implies

$$\begin{aligned} 0 &= \int_{\mathcal{C}_1} f(z)dz - \int_{\mathcal{C}_2} f(z)dz \\ &= \int_{\mathcal{C}_1} f(z)dz + \int_{-\mathcal{C}_2} f(z)dz \\ &= \int_{\mathcal{C}} f(z)dz, \text{ where } \mathcal{C} = \mathcal{C}_1 + (-\mathcal{C}_2) \end{aligned}$$

which shows that the integral of $f(z)$ around closed contours lying in D is 0.

(c) $\Rightarrow$ **(a)** Suppose integrals of $f(z)$ around closed contours in D have a value of 0. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote any two contours in D from point z_1 to a point z_2. Then $\mathcal{C} = \mathcal{C}_1 + (\mathcal{C}_2)$ is a closed contour in D and by assumption,

$$\begin{aligned} 0 &= \int_{\mathcal{C}} f(z)dz \\ &= \int_{\mathcal{C}_1+(\mathcal{C}_2)} f(z)dz \\ &= \int_{\mathcal{C}_1} f(z)dz + \int_{-\mathcal{C}_2} f(z)dz \\ &= \int_{\mathcal{C}_1} f(z)dz - \int_{\mathcal{C}_2} f(z)dz \\ &\Rightarrow \int_{\mathcal{C}_1} f(z)dz \\ &= \int_{\mathcal{C}_2} f(z)dz \end{aligned}$$

Note that this shows **(c)** $\Rightarrow$ **(b)**. Now fix any $z_0 \in D$ and define a function $F : D \to \mathbb{C}$ by

$$F(z) = \int_{z_0}^{z} f(s)ds, \quad \forall z \in D$$

The path independence of integrals imply that F is well-defined.

Claim:

$$F'(z) = f(z) \quad \forall z \in D$$

Let $z + \Delta z$ be any point distinct from z and lying in some neighborhood of z that is small enough to be contained in D (such a neighborhood exists since D

is an open set).

$$\begin{aligned} F(z+\Delta z)-F(z) &= \int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds \\ &= \int_{z_0}^{z} f(s)ds + \int_{z}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds \\ &= \int_{z}^{z+\Delta z} f(s)ds \end{aligned}$$

Which implies

$$\begin{aligned} \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s)ds - f(z) \\ &= \frac{\int_{z}^{z+\Delta z} f(s)ds - f(z)\Delta z}{\Delta z} \end{aligned}$$

We have $\int_z^{z+\Delta z} ds = [s]_z^{z+\Delta z} = (z+\Delta z) - z = \Delta z$, which gives us

$$\begin{aligned} \frac{F(z+\Delta z)-F(z)}{\Delta z} &= \frac{\int_{z}^{z+\Delta z} f(s)ds - f(z)\int_{z}^{z+\Delta z} ds}{\Delta z} \\ &= \frac{\int_{z}^{z+\Delta z} (f(s)-f(z))\, ds}{\Delta z} \end{aligned}$$

Since f is continuous at z, then for any $\epsilon > 0$ we may choose $\delta > 0$ such that, if $0 < |\Delta z| < \delta$, then $|f(s) - f(z)| < \epsilon$ for all s on the line segment from z to $z + \Delta z$. So, for $0 < |\Delta z| < \delta$,

$$\begin{aligned} \left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| &= \left|\frac{\int_{z}^{z+\Delta z} ((f(s)-f(z))\, ds}{\Delta z}\right| \\ &\leq \frac{\epsilon|\Delta z|}{|\Delta z|} \\ &= \epsilon \end{aligned}$$

By ML-estimate with $M = \epsilon, L = \Delta z$, by assuming we are using a line from z to $z + \Delta z$,

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$$

which proves the claim, and so proves **(a)**. □

Green's Theorem

Let $\mathcal{C}$ be a positively oriented, piecewise smooth simple closed curve that bounds a domain D in the complex plane. Let $P(x, y)$ and $Q(x, y)$ be two real-valued functions defined on an open set R that contains D, and suppose that P and Q have continuous first order partial derivatives on R, then

$$\int_{\mathcal{C}} P_{dx} + Q_{dx} = \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$$

Cauchy's Theorem

Suppose that D is a simply connected domain, $f : 0 \to \mathbb{C}$ is analytic in D, and that f' is continuous in D. Then, for every simple closed contour $\mathcal{C}$ in D,

$$\int_{\mathcal{C}} f(z)dz = 0$$

Proof:
The proof of this theorem follows immediately from Green's Theorem in $\mathbb{R}^2$ and the Cauchy-Riemann equations. Recall:

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}} (udx - vdy) + i \int_{\mathcal{C}} (vdx + udy)$$

Now we have assumed that f' is continuous in D. As a consequence, the real and imaginary parts of $f(z) = u + iv$ and their first partial derivatives are continuous in D. By Green's Theorem, we obtain

$$\begin{aligned} \int_{\mathcal{C}} udx - vdy &= \iint_D \left(-\frac{dv}{dx} - \frac{du}{dy} \right) dA \\ \int_{\mathcal{C}} vdx + udy &= \iint_D \left(-\frac{du}{dx} - \frac{dv}{dy} \right) dA \end{aligned}$$

Which implies

$$\begin{aligned} \int_{\mathcal{C}} f(z)dz = &\iint_D \left(-\frac{dv}{dx} - \frac{du}{dy} \right) dA \\ &+ i \iint_D \left(-\frac{du}{dx} - \frac{dv}{dy} \right) dA \end{aligned}$$

Since f is analytic in D, the real and imaginary parts of f, u, and v respectively, satisfy the C-R equations: $\frac{du}{dx} = \frac{dv}{du}$ and $\frac{du}{dy} = -\frac{dv}{dx}$ at every point in D. Therefore,

$$\int_{\mathcal{C}} f(z)dz = 0$$

4.46.A (Cauchy-Goursat Theorem)

If a function f is analytic at all points interior to and on a simple closed contour $\mathcal{C}$,

$$\int_{\mathcal{C}} f(z)dz = 0$$

Proof:
See Sec. 4.47.

4.48.A (Cauchy-Goursat Theorem Extended)

We can remove the "simple" requirement of the contour in the original Cauchy-Goursat Theorem by using simply connected domains. This a more general version of the theorem since it allows the curve to cross itself.

If D is a simply connected domain and $f : D \to \mathbb{C}$ is analytic in D, then

$$\int_{\mathcal{C}} f(z)dz = 0$$

for any closed (not necessarily simple) contour $\mathcal{C}$ lying in D.

Proof:
Case 1: $\mathcal{C}$ is a simple closed contour.
If $\mathcal{C}$ is simple and closed then the region enclosed by $\mathcal{C}$ is contained in D, and f is analytic in the region enclosed by $\mathcal{C}$ and on $\mathcal{C}$. By the version of Cauchy-Goursat in Theorem 4.46.A, we have $\int_{\mathcal{C}} f(z)dz = 0$.

Case 2: $\mathcal{C}$ is not simple, but intersects itself a finite number of times.
Let $\mathcal{C}$ be such a contour in D. Subdivide $\mathcal{C}$ into a finite number of simple closed contours. Then,

$$\int_{\mathcal{C}} f(z)dz = \sum_i (\pm 1) \int_{\mathcal{C}_i} f(z)dz$$

where each contour $\mathcal{C}_i$ is simple and closed. Therefore,

$$\int_{\mathcal{C}_i} f(z)dz = 0, \quad \forall i$$
$$\sum_i (\pm 1) \int_{\mathcal{C}_i} f(z)dz = 0$$

hence $\int_{\mathcal{C}} f(z)dz = 0$. $\square$

Theorem 4.44.A

In a simple connected domain, any analytic function has an antiderivative, its contour integrals are independent of the path, and its integrals over a closed contour equal 0.

Theorem 4.49.A (Principle of Deformation of Contours)

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote positively orient simple closed contours, where $\mathcal{C}_1$ is interior to $\mathcal{C}_2$. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{\mathcal{C}_2} f(z)dz = \int_{\mathcal{C}_1} f(z)dz$$

This allows us to evaluate an integral over a complicated simple closed contour $\mathcal{C}$ by replacing $\mathcal{C}$ with a contour $\mathcal{C}_1$ that is more convenient.

For example, consider integrating over a function with a singularity. We can obtain a general solution to these types of problems:

$$\int_{\mathcal{C}} \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

where $z_0 \in \mathbb{C}$ is a constant interior to any simple closed contour $\mathcal{C}$, and $n \in \mathbb{Z}$.

Cauchy-Goursat Theorem (for Multiply Connected Domains)

Let $\mathcal{C}$ be a simple closed contour within a domain D, and let C_k, where $k = 1, 2, \ldots, n$ be disjoin simple closed contours interior to the contour $\mathcal{C}$. If $f(z)$ is analytic at all points inside or on $\mathcal{C}$, and outside or on each $\mathcal{C}_k$, then

$$\int_{\mathcal{C}} f(z)dz = \sum_{k=1}^{n} \int_{\mathcal{C}_k} f(z)dz$$

Proof:
The idea is to use a crosscut between $\mathcal{C}$ and $\mathcal{C}_k, \quad \forall k = 1, 2, \ldots, n$. This produces a simply connected domain. Then as in the case of the doubly connected domain, we have

$$\int_{\mathcal{C}} f(z)dz = \sum_{k=1}^{n} \int_{\mathcal{C}_k} f(z)dz$$

□

Remarks 3

The Cauchy-Goursat Theorem **gives only sufficient condition for the integral** $\int_{\mathcal{C}} f(z)dz$ **to be zero** (namely, f is analytic inside $\mathcal{C}$ and on $\mathcal{C}$). However, in certain cases, $\int_{\mathcal{C}} f(z)dz = 0$ even if $f(z)$ is not analytic inside $\mathcal{C}$ and on $\mathcal{C}$. For example, $f(z) = \frac{1}{z^2}$ is not analytic inside the contour

$$\mathcal{C} : \gamma(t) = Re^{it}, \quad 0 \leq 2\pi, R > 0$$

yet,

$$\int_{\mathcal{C}} \frac{1}{z^2} dz = 0$$

Theorem 4.50.A (Cauchy's Integral Formula)

Let $f(z)$ be an analytic function in a simply connected domain D. If $\mathcal{C}$ is a simple closed contour that lies inside D, and if z_0 is a fixed point inside $\mathcal{C}$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz$$

Proof:
Let D be a simply connected domain, $\mathcal{C}$ a simple contour inside D, and z_0 an interior point of $\mathcal{C}$. Let $\mathcal{C}_r$ be a circle centered at z_0 and radius r small enough so that $\mathcal{C}_r$ lies within the interior of $\mathcal{C}$. By the principle of deformation of contours, we have

$$\int_{\mathcal{C}} \frac{f(z)}{z-z_0} dz = \int_{\mathcal{C}_r} \frac{f(z)}{z-z_0} dz$$

We want to show that the value of the integral on the RHS is $2\pi i f(z_0)$. To do this, we add and subtract the constant $f(z_0)$ in the numerator of the integral:

$$\begin{aligned}\int_{\mathcal{C}_r} \frac{f(z)}{z-z_0} dz &= \int_{\mathcal{C}_r} \frac{f(z)-f(z_0)+f(z_0)}{z-z_0} dz \\ &= \int_{\mathcal{C}_r} \frac{f(z)-f(z_0)}{z-z_0} dz + f(z_0) \int_{\mathcal{C}_r} \frac{1}{z-z_0} dz\end{aligned}$$

Now, let $\mathcal{C}_r$ be parametrized by

$$\begin{aligned}\gamma(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi \\ \int_{\mathcal{C}_r} \frac{1}{z-z_0} = \int_0^{2\pi} \frac{rie^{it}}{re^it} dt = \int_0^{2\pi} idt = 2\pi i \\ \Rightarrow \int_{\mathcal{C}_r} \frac{f(z)}{z-z_0} dz = \int_{\mathcal{C}_r} \frac{f(z)-f(z_0)}{z-z_0} dz + 2\pi i f(z_0)\end{aligned}$$

Since f is continuous at z_0, then for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. In particular, if we choose the circle $\mathcal{C}_r$ to have a radius $r = \frac{1}{2}\delta < \delta$, then by the ML-inequality, we have

$$\left| \int_{\mathcal{C}_r} \frac{f(z)-f(z_0)}{z-z_0} dz \right| \le \frac{\epsilon}{r}(2\pi r) = 2\pi\epsilon$$

Thus the absolute value of the integral can be made arbitrarily small.

$$\begin{aligned}\int_{\mathcal{C}_r} \frac{f(z)}{z-z_0} dz &= \int_{\mathcal{C}_r} \frac{f(z)-f(z_0)}{z-z_0} dz + 2\pi i f(z_0) \\ &\Rightarrow \int_{\mathcal{C}_r} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \\ &= \int_{\mathcal{C}_r} \frac{f(z)-f(z_0)}{z-z_0} dz \\ &\Rightarrow \left| \int_{\mathcal{C}_r} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| \\ &= \left| \int_{\mathcal{C}_r} \frac{f(z)-f(z_0)}{z-z_0} dz \right| \\ &\le 2\pi\epsilon\end{aligned}$$

Divide both sides by $\frac{1}{2\pi}$,

$$\left|\frac{1}{2\pi i}\int_{\mathcal{C}_r}\frac{f(z)}{z-z_0}dz - f(z_0)\right| \leq \epsilon$$

Since ϵ is arbitrary,

$$\frac{1}{2\pi i}\int_{\mathcal{C}_r}\frac{f(z)}{z-z_0}dz - f(z_0) = 0$$

Since

$$\frac{1}{2\pi i}\int_{\mathcal{C}_r}\frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f(z)}{z-z_0}dz$$
$$f(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f(z)}{z-z_0}dz$$

□

Theorem 4.50.A (Cauchy's Integral Formula - Version 2)

If f is analytic at all points within and on a simple closed contour $\mathcal{C}$, and z_0 is any point interior to $\mathcal{C}$, then

$$f(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f(z)}{z-z_0}dz$$

Theorem (?)

Let $f(z)$ and $g(z)$ be two complex functions that are analytic inside and on a simple closed contour $\mathcal{C}$. If $f(z) = g(z)$ for all $z \in \mathcal{C}$, then $f(z) = g(z)$ for all z interior to $\mathcal{C}$.

Proof:
Let z_0 be a point inside $\mathcal{C}$. Then, by Cauchy's Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f(z)}{z-z_0}dz$$
$$g(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{g(z)}{z-z_0}dz$$

Let $\gamma(t)$, $a \leq t \leq b$ be a parametric representation of the contour $\mathcal{C}$. Then,

$$\int_{\rfloor}\frac{f(z)}{z-z_0} = \int_a^b \frac{f(\gamma(t))}{\gamma(t)-z_0}\gamma'(t)dt$$
$$\int_{\rfloor}\frac{g(z)}{z-z_0} = \int_a^b \frac{g(\gamma(t))}{\gamma(t)-z_0}\gamma'(t)dt$$

But $f(\gamma(t)) = g(\gamma(t))$ for all $t \in [a, b]$, therefore

$$\begin{aligned}\int_a^b \frac{f(\gamma(t))}{\gamma(t) - z_0}\gamma'(t)dt &= \int_a^b \frac{g(\gamma(t))}{\gamma(t) - z_0}\gamma'(t)dt \\ &\Rightarrow \int_{\mathcal{C}} \frac{f(z)}{z - z_0}dz = \int_{\mathcal{C}} \frac{g(z)}{z - z_0} \\ &\Rightarrow 2\pi i f(z_0) = 2\pi i g(z_0) \\ &\Rightarrow f(z_0) = g(z_0)\end{aligned}$$

Since z_0 is arbitrary, $f(z) = g(z)$ for all z interior to $\mathcal{C}$. $\square$

Theorem 4.51.A (Cauchy's Integral Formula for Derivatives

Suppose that f is analytic in a simply connected domain D and $\mathcal{C}$ is any simple closed contour lying entirely inside D. Then for any point z_0 inside $\mathcal{C}$, $f^{(n)}(z_0)$ exists and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}}dz$$

where $n = 0, 1, 2, 3, \ldots$.

Proof:
Let z be any point inside $\mathcal{C}$, then by Cauchy's Integral Formula we have

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(s)}{s - z}ds$$

then,

$$\begin{aligned}f'(z) &= \frac{1}{2\pi i}\frac{d}{dz}\left(\int_{\mathcal{C}} \frac{f(s)}{s - z}ds\right) \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{d}{dz}\left(\frac{f(s)}{s - z}\right)ds \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} \left[f(s)\frac{d}{dz}(\frac{1}{s - z})\right]ds \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(s)}{(s - z)^2}ds\end{aligned}$$

Differentiating again w.r.t. z gives us,

$$\begin{aligned}f''(z) &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{d}{dz}\left(\frac{f(s)}{(s - z)^2}\right)ds \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{2f(s)}{(s - z)^2}ds \\ &= \frac{2!}{2\pi i}\int_{\mathcal{C}} \frac{f(s)}{(s - z)^2}ds\end{aligned}$$

With repeated differentiation we see that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{f(s)}{(s-z)^{n+1}} ds$$

$\square$

Theorem (??)

If f is analytic in a simple connected domain D, then f has derivatives of all orders at every point z in D. Furthermore, $f^{(n)}$, for $n = 0, 1, 2, \ldots$ are analytic in D.

Proof:
If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in D, we have shown its derivatives of all orders exists at any point z in D and so $f', f'', \ldots$ are continuous.

$$f'(z) = \frac{du}{dx} + i\frac{dv}{dx} = \frac{dv}{dy} - i\frac{du}{dy}$$

$$f''(z) = \frac{d^2u}{dx^2} + i\frac{d^2v}{dx^2} = \frac{d^2v}{dydx} - i\frac{d^2u}{dydx}$$

$$\vdots$$

Thus the real functions u and v have continuous partial derivatives of all orders and satisfy the $C - R$ equations at any point of D. Hence $f^{(n)}(z)$ is analytic for $n = 1, 2, 3, \ldots$ and all $z \in D$. $\square$

Theorem (???)

Let $f(z)$ be analytic inside and on a simple closed contour $\mathbb{C}$, then we have

$$\int_{\mathcal{C}} \frac{f^{(n)}(z)}{z - z_0} dz = n! \int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

for $n = 1, 2, 3, \ldots$.

Proof:
From CIF applied to $g(z) = f^{(n)}(z)$ for a fixed $n \in \mathbb{N}$, we have

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f^{(n)(z)}}{z - z_0} dz$$

From CIFD applied to $f(z)$ and n, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f^{(n)}(z)}{z - z_0} dz = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\Rightarrow \int_{\mathcal{C}} \frac{f^{(n)}(z)}{z - z_0} dz = n! \int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$\square$

Theorem 4.52.1

If a function f is analytic at a given point, then its derivatives of all orders are analytic at that point too.

Proof:
See notes (pg. 170). $\square$

Corollary 4.52.A

If $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z_0 = x_0 + iy_0$, then the real and imaginary parts $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of all orders at (x_0, y_0).

Proof:
See notes (pg. 172). $\square$

Corollary 4.52.2 (Morera's Theorem)

If $f : D \to \mathbb{C}$ is continuous in a domain D and $\int_{\mathcal{C}} f(z)dz = 0$ for all closed contours $\mathcal{C}$ in D, then f is analytic in D, and $f(z) = F'(z)$ for some analytic function F on D.

Proof:
See notes (pg. 173). $\square$

Theorem ????

Let $f : D \to \mathbb{C}$ be continuous in a simply connected domain and let $\mathcal{C}$ be closed contour in D. Then a necessary and sufficient condition for f to be analytic in D is that $\int_{\mathcal{C}} f(z)dz = 0$.

Proof:
Excersise. $\square$

Theorem 4.52.3 (Cauchy's Inequality)

Suppose that a function f is analytic on and inside a circle $\mathcal{C}_R$ centered at point z_0 with radius R. If M is the maximum value of $|f(z)|$ on $\mathcal{C}_R$, then for all $n \in \mathbb{N}$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{R^n}$$

Proof:
See notes (pg. 175). $\square$

Theorem 4.53.1 (Liouville's Theorem)

If f is entire and there is a constant M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is constant over the complex plane.

Proof:
See notes (pg. 176). $\square$

Remarks 4

$f(z) = \sin z$ and $f(z) = \cos z$ are entire functions and therefore they are unbounded. This is in sharp to the bounded real functions $f(x) = \sin x$ and $f(x) = \cos x$.

Theorem 4.53.2 (Fundamental Theorem of Algebra)

Any nonconstant complex polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, when $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and of degree $n \geq 1$, has at least one zero. That is, there exists at least one point $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof:
See notes (pg. 178). $\square$

Remarks 5

Consider a polynomial $p(z)$ of degree $n \geq 1$ with complex coefficients. By the Fundamental Theorem of Algebra, $p(z)$ has a zero, say $z_1 \in \mathbb{C}$. So by the Factor Theorem, $p(z) = (z - z_1)q_1(z)$ for some complex polynomial $q_1(z)$ of degree $n - 1$. We can repeat this argument with $q_1(z)$ and obtain $p(z) = (z - z_1)(z - z_2)q_2(z)$, and so forth. Then we can factor $p(z)$ into linear factors of the form $p(z) = c(z - z_1)(z - z_2)\ldots(z - z_n)$, where c is a nonzero complex number and $z_1, \ldots, z_n \in \mathbb{C}$ (z_i's might not be distinct).